Well-posedness and spectral representation of linear initial-boundary value problems

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Outline

Initial-boundary value problems

Consider the following IBVPs for linearized KdV:

Coupled problem

$$
\begin{aligned}
& & [\partial_t + \partial_x^3]q(x,t) &= 0, & (x,t) &\in (0,1)\times(0,T),\\
&\text{where} & q(x,0) &= q_0(x), & x &\in [0,1]\\
&\text{and} & q(0,t)=0, \quad q(1,t)=0, \quad q_x(1,t) &= q_x(0,t)/2, & t &\in [0,T].
\end{aligned}
$$

Uncoupled problem

$$
\begin{aligned}
& & [\partial_t + \partial_x^3]q(x,t) &= 0, & (x,t) &\in (0,1)\times(0,T),\\
&\text{where} & q(x,0) &= q_0(x), & x &\in [0,1]\\
&\text{and} & q(0,t)=0, \quad q(1,t)=0, \quad q_x(1,t) &= 0, & t &\in [0,T].
\end{aligned}
$$

Initial datum $q_0 \in C^\infty[0,1]$ compatible with boundary conditions.

Initial-boundary value problems

General problem

$$[\partial_t + a(-i\partial_x)^n]q(x,t) = 0, \qquad (x,t) \in (0,1)\times(0,T),$$

where

$$q(x,0) = q_0(x), \qquad x \in [0,1]$$

and

$$\sum_{r=0}^{n-1}\left[b_{r\,j}\partial_x^r q(0,t) + \beta_{r\,j}\partial_x^r q(1,t)\right] = h_j, \qquad t \in [0,T]$$

for $j = 1, 2, \dots, n$.
Initial datum $q_0 \in C^\infty[0,1]$ compatible with boundary conditions,
boundary data $h_j \in C^\infty[0,T]$,
boundary coefficients $b, \beta \in \mathbb{C}^{n\times n}$ such that $(b|\beta)$ full rank.

If n odd then $a = \pm i$.
If n even then $\mathrm{Re}(a) \geqslant 0$.

Solution step 1: Implicit solution

If q satisfies PDE on domain $(0,1) \times (0,T)$ then

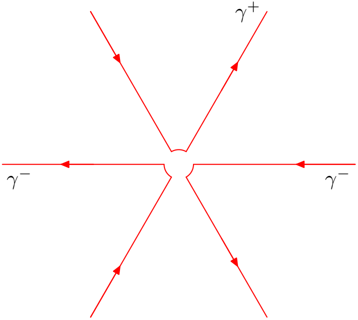
$$2\pi q(x,t) = \int_{\mathbb{R}} e^{i\lambda x + i\lambda^3 t} \hat{q}_0(\lambda) d\lambda - \int_{\gamma^+} e^{i\lambda x + i\lambda^3 t} \sum_{r=0}^{2} c_r(\lambda) Q_r(0,\lambda) d\lambda - \int_{\gamma^-} e^{i\lambda(x-1) + i\lambda^3 t} \sum_{r=0}^{2} c_r(\lambda) Q_r(1,\lambda) d\lambda,$$

where

$$\hat{q}_0(\lambda) = \int_0^1 e^{-i\lambda x} q_0(x) dx,$$

$$Q_r(X,\lambda) = \int_0^T e^{-i\lambda^3 t} \partial_x^r q(X,t) dt,$$

$$c_r(\lambda) = \text{monomial}.$$



Obtained via:

- Fourier transform (simpler)
- Lax pair & Riemann-Hilbert formalism (more complex but generalizes to nonlinear equations)

Solution step 2: Global relation

The transformed boundary functions $Q_r(0,\lambda)$, $Q_r(1,\lambda)$ must satisfy

$$\sum_{r=0}^{2} c_r(\lambda)\left[Q_r(0,\lambda) - e^{-i\lambda}Q_r(1,\lambda)\right] = \hat{q}_0(\lambda) - e^{-i\lambda^3 T}\hat{q}_T(\lambda), \qquad \forall\ \lambda \in \mathbb{C},$$

where

$$\hat{q}_T(\lambda) = \int_0^1 e^{-i\lambda x} q(x,T)dx, \qquad Q_r(X,\lambda) = \int_0^T e^{-i\lambda^3 t}\partial_x^r q(X,t)dt.$$

Obtained via Green's theorem applied to PDE on (x,t) rectangle $[0,1]\times[0,T]$.

Solution step 2: Global relation

The transformed boundary functions $Q_r(0,\lambda)$, $Q_r(1,\lambda)$ must satisfy

$$\sum_{r=0}^{2} c_r(\lambda\quad)\left[Q_r(0,\lambda) - e^{-i\lambda}\quad Q_r(1,\lambda)\right] = \hat{q}_0(\lambda\quad) - e^{-i\lambda^3 T}\hat{q}_T(\lambda\quad), \qquad \forall\ \lambda \in \mathbb{C},$$

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Used to construct Dirichlet-to-Neumann map for this IBVP:

Solution step 2: Global relation

The transformed boundary functions $Q_r(0,\lambda)$, $Q_r(1,\lambda)$ must satisfy

$$\sum_{r=0}^{2} c_r(\lambda\omega^j)\left[Q_r(0,\lambda) - e^{-i\lambda\omega^j}Q_r(1,\lambda)\right] = \hat{q}_0(\lambda\omega^j) - e^{-i\lambda^3 T}\hat{q}_T(\lambda\omega^j), \qquad \forall\ \lambda\in\mathbb{C}, \quad j=0,1,2,$$

where $\omega = e^{2\pi i/3}$,

$$\hat{q}_T(\lambda) = \int_0^1 e^{-i\lambda x} q(x,T)dx, \qquad Q_r(X,\lambda) = \int_0^T e^{-i\lambda^3 t}\partial_x^r q(X,t)dt.$$

Obtained via Green's theorem applied to PDE on (x,t) rectangle $[0,1]\times[0,T]$.

Used to construct Dirichlet-to-Neumann map for this IBVP:

- System of 3 equations in 6 unknowns ($\hat{q}_0$ is known, treat $\hat{q}_T$ as known for now).

Solution step 2: Global relation

The transformed boundary functions $Q_r(0,\lambda)$, $Q_r(1,\lambda)$ must satisfy

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Obtained via Green's theorem applied to PDE on (x,t) rectangle $[0,1]\times[0,T]$.

Used to construct Dirichlet-to-Neumann map for this IBVP:

- System of 3 equations in 6 unknowns ($\hat{q}_0$ is known, treat $\hat{q}_T$ as known for now).
- Boundary conditions provide another 3 equations.

Solution step 2: Dirichlet-to-Neumann Map

Solving the system yields

$$2\pi q(x,t) = \int_{\mathbb{R}} e^{i\lambda x + i\lambda^3 t}\hat{q}_0(\lambda)d\lambda - \int_{\gamma^+} e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda; q_0)}{\Delta(\lambda)}d\lambda - \int_{\gamma^-} e^{i\lambda(x-1) + i\lambda^3 t}\frac{\zeta^-(\lambda; q_0)}{\Delta(\lambda)}d\lambda$$
$$+ \int_{\gamma^+} e^{i\lambda x + i\lambda^3 t}\frac{e^{-i\lambda^3 T}\zeta^+(\lambda; q(\cdot, T))}{\Delta(\lambda)}d\lambda + \int_{\gamma^-} e^{i\lambda(x-1) + i\lambda^3 t}\frac{e^{-i\lambda^3 T}\zeta^-(\lambda; q(\cdot, T))}{\Delta(\lambda)}d\lambda,$$

where

$\Delta(\lambda)$ is	an exponential polynomial in λ,
$\zeta^{\pm}(\lambda; f)$ is	a sum of exponential polynomials, each multiplied by Fourier transform $\hat{f}(\lambda\omega^r)$,

depend upon the boundary conditions.

Solution step 2: Dirichlet-to-Neumann Map

Coupled problem

$$\Delta(\lambda) = e^{i\lambda} + \omega e^{i\omega\lambda} + \omega^2 e^{i\omega^2\lambda} + 2(e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda}),$$

$$\zeta^+(\lambda; f) = \hat{f}(\lambda)(e^{i\lambda} + 2\omega e^{-i\omega\lambda} + 2\omega^2 e^{-i\omega^2\lambda}) + \hat{f}(\omega\lambda)(\omega e^{i\omega\lambda} - 2\omega e^{-i\lambda}) + \hat{f}(\omega^2\lambda)(\omega^2 e^{i\omega^2\lambda} - 2\omega^2 e^{-i\lambda}),$$

$$\zeta^-(\lambda; f) = -\hat{f}(\lambda)(2 + \omega^2 e^{-i\omega\lambda} + \omega e^{-i\omega^2\lambda}) - \omega\hat{f}(\omega\lambda)(2 - e^{-i\omega^2\lambda}) - \omega^2\hat{f}(\omega^2\lambda)(2 - e^{-i\omega\lambda}).$$

Uncoupled problem

$$\Delta(\lambda) = e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda},$$

$$\zeta^+(\lambda; f) = \hat{f}(\lambda)(\omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda}) - (\omega\hat{f}(\omega\lambda) + \omega^2\hat{f}(\omega^2\lambda))e^{-i\lambda},$$

$$\zeta^-(\lambda; f) = -\hat{f}(\lambda) - \omega\hat{f}(\omega\lambda) - \omega^2\hat{f}(\omega^2\lambda).$$

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Solving the system yields

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where

$\Delta(\lambda)$ is	an exponential polynomial in λ,
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depend upon the boundary conditions.

Have to remove dependence on *unknown* $q(\cdot, T)$ to get explicit representation of solution.

Solution step 2: Contour deformation

Dirichlet-to-Neumann map yields

$$2\pi q(x,t) = \ldots + \int_{\gamma^+} e^{i\lambda x - i\lambda^3(T-t)} \frac{\zeta^+(\lambda; q(\cdot,T))}{\Delta(\lambda)} d\lambda + \int_{\gamma^-} e^{i\lambda(x-1) - i\lambda^3(T-t)} \frac{\zeta^-(\lambda; q(\cdot,T))}{\Delta(\lambda)} d\lambda$$

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Exponential in λ with x, t parameters

$$\frac{\text{Exp poly with FT of function at final time}}{\text{Exponential polynomial}}$$

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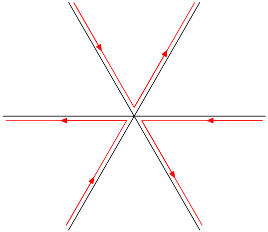

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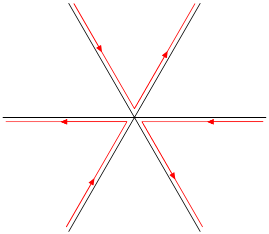

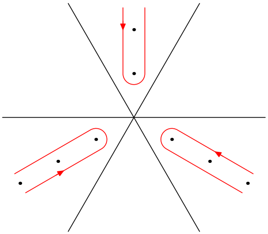

Jordan

Solution step 2: Contour deformation

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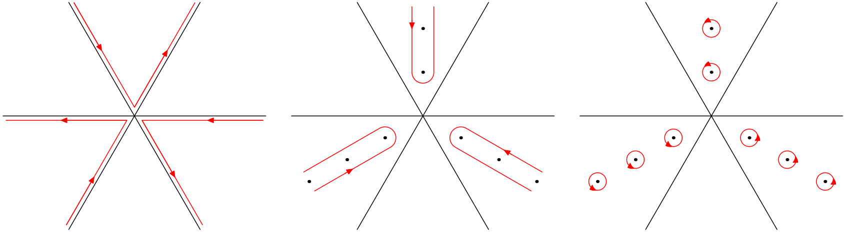

Jordan Cauchy

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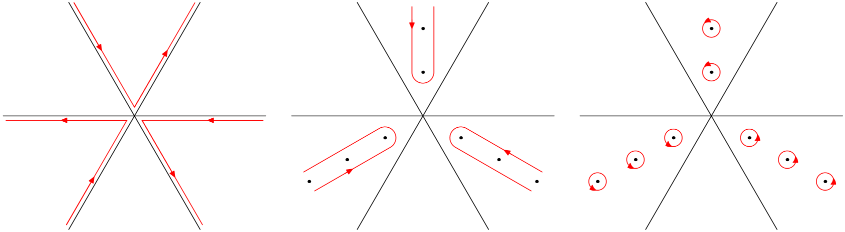

Jordan   Cauchy

$$= -\sum_{\sigma} \operatorname*{Res}_{\lambda=\sigma} \left(e^{i\lambda x - i\lambda^3(T-t)} \frac{\zeta^+(\lambda; q(\cdot, T))}{\Delta(\lambda)} \right)$$

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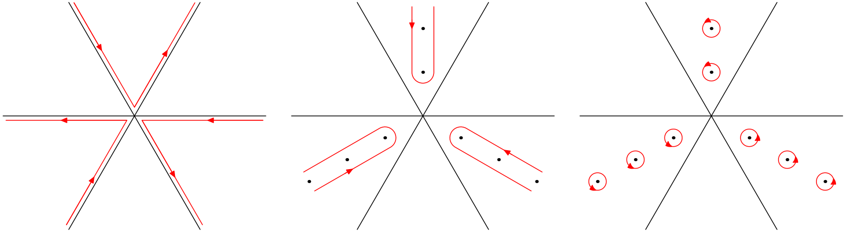

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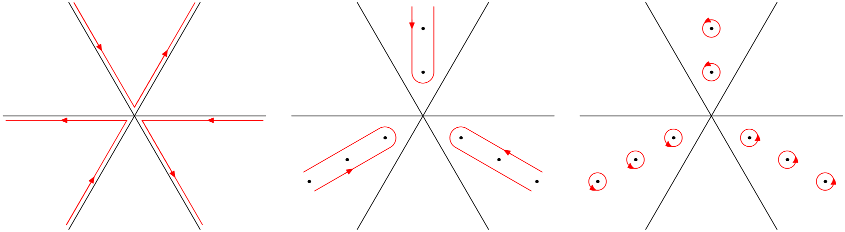

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Requirements for deformation:

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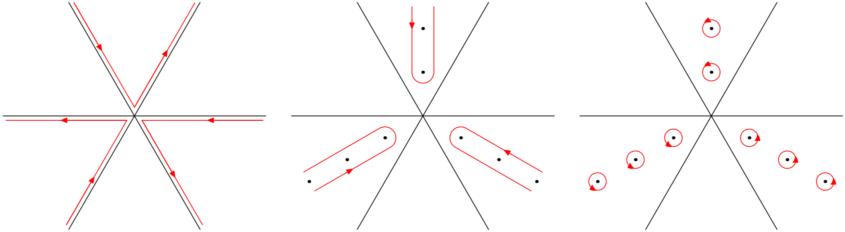

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Requirements for deformation:

- Only isolated poles, analytic elsewhere

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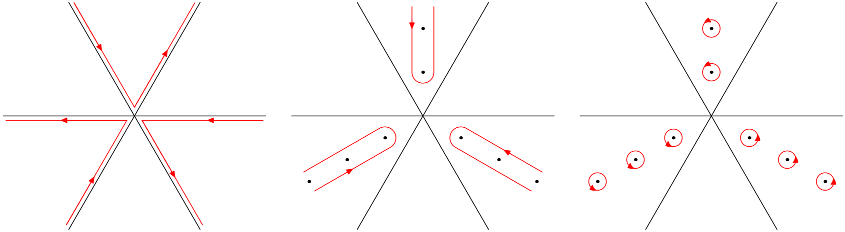

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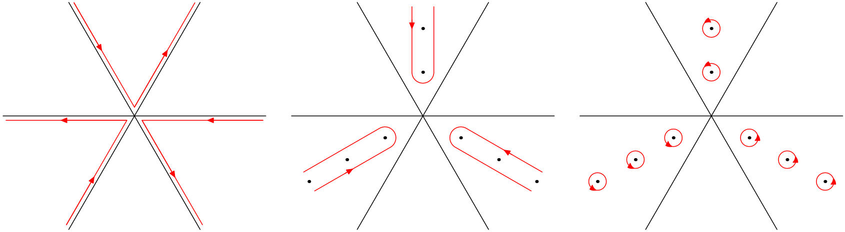

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- Decays at infinity inside contour $\gamma^\pm$

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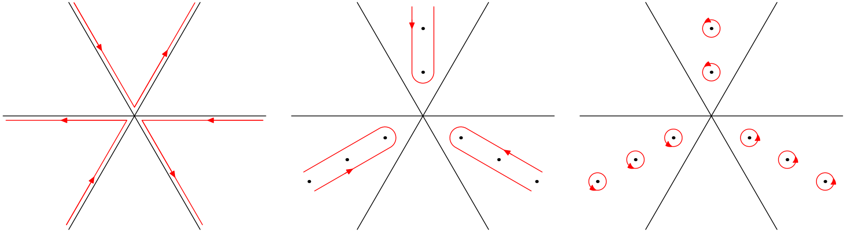

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Requirements for deformation:

- Only isolated poles, analytic elsewhere ✓
- Decays at infinity inside contour $\gamma^{\pm}$?

Well-posedness

Theorem

IBVP is well-posed iff both

1. *the meromorphic functions*

$$\frac{\zeta^{\pm}(\lambda; q(\cdot, T))}{\Delta(\lambda)}$$

   decay as $\lambda \to \infty$ within the sectors enclosed in $\gamma^{\pm}$, away from any zeros of Δ.

2. *the zeros of Δ lie asymptotically on $\gamma^{\pm}$ and/or outside sectors bounded by $\gamma^{\pm}$.*

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1. the contour deformation is possible.
2. the resulting series of residues converges.

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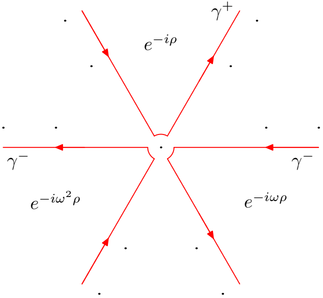
which dominates $\zeta^{\pm}$ in each sector.
Series has no terms.



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Conditions guarantee:

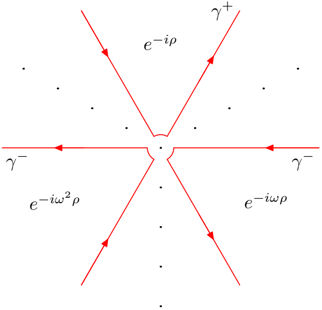
1. the contour deformation is possible.
2. the resulting series of residues converges.

Both problems well-posed:

uncoupled problem

$$\Delta(\lambda) = e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda},$$

which dominates $\zeta^{\pm}$ in each sector.
Series has no terms.



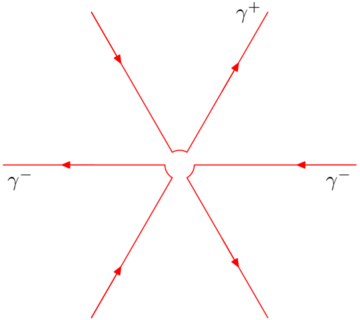
Representations of solution: contour integral

Explicit *integral representation* of solution:

$$2\pi q(x,t) = \int_{\mathbb{R}} e^{i\lambda x + i\lambda^3 t} \hat{q}_0(\lambda) d\lambda - \int_{\gamma^+} e^{i\lambda x + i\lambda^3 t} \frac{\zeta^+(\lambda; q_0)}{\Delta(\lambda)} d\lambda - \int_{\gamma^-} e^{i\lambda(x-1) + i\lambda^3 t} \frac{\zeta^-(\lambda; q_0)}{\Delta(\lambda)} d\lambda$$

$$+ \int_{\gamma^+} e^{i\lambda x + i\lambda^3 t} \frac{e^{-i\lambda^3 T} \zeta^+(\lambda; q(\cdot, T))}{\Delta(\lambda)} d\lambda + \int_{\gamma^-} e^{i\lambda(x-1) + i\lambda^3 t} \frac{e^{-i\lambda^3 T} \zeta^-(\lambda; q(\cdot, T))}{\Delta(\lambda)} d\lambda$$

with contours



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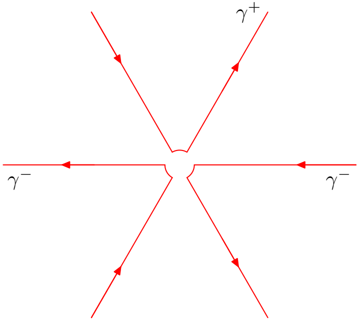

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$$= \left(\int_{\Gamma_0} + \int_{\Gamma^+}\right) e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda;q_0)}{\Delta(\lambda)}d\lambda + \int_{\Gamma^-} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda;q_0)}{\Delta(\lambda)}d\lambda,$$

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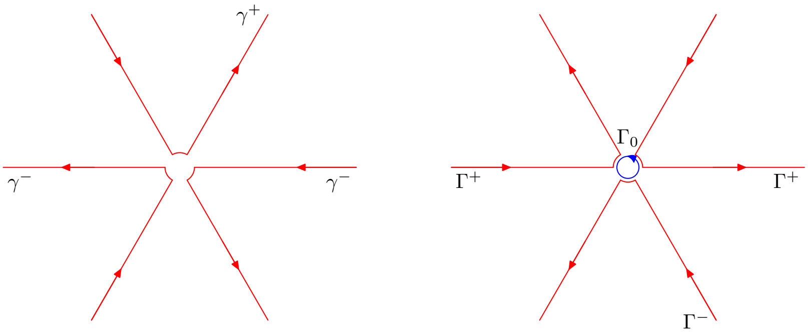
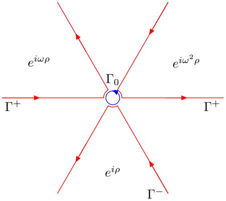


Representations of solution: discrete series

We can write a *series representation* of solution:

$$2\pi q(x,t) = \left(\int_{\Gamma_0} + \int_{\Gamma^+}\right) e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda;q_0)}{\Delta(\lambda)}d\lambda + \int_{\Gamma^-} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda;q_0)}{\Delta(\lambda)}d\lambda$$
$$= \sum_{\sigma\in\overline{\mathbb{C}^+}} 2\pi i \operatorname*{Res}_{\lambda=\sigma} e^{i\lambda x+i\lambda^3 t}\frac{\zeta^+(\lambda;q_0)}{\Delta(\lambda)} + \sum_{\sigma\in\mathbb{C}^-} 2\pi i \operatorname*{Res}_{\lambda=\sigma} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda;q_0)}{\Delta(\lambda)},$$

iff it is possible to 'close' the contours $\Gamma^\pm$ onto circles about each pole.



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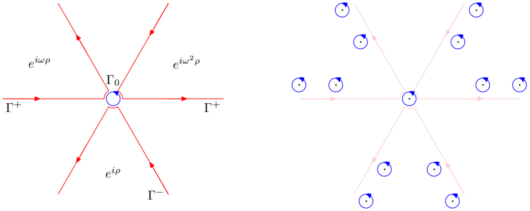

$$\Delta(\lambda) = e^{i\lambda} + \omega e^{i\omega\lambda} + \omega^2 e^{i\omega^2\lambda} + 2(e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda}),$$

which dominates $\zeta^\pm$ in each sector.

Series representation *valid for coupled problem*.

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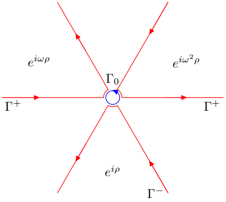
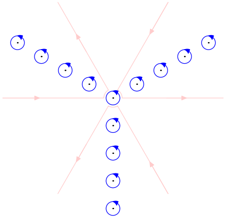
uncoupled





$$\Delta(\lambda) = e^{-i\lambda} + \omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda},$$

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Series representation *invalid for uncoupled problem*.

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\end{aligned}
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iff it is possible to 'close' the contours $\Gamma^\pm$ onto circles about each pole.

Series representation valid for coupled problem but invalid for uncoupled problem.

Easily-checked conditions for well-posedness and existence of series representation of solution have been identified for general IBVP.

Outline

Transform pair

Having established well-posedness, we can view the unified method as a standard (transform, inverse transform) pair tailored for the IBVP:

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$$f(x) \mapsto F(\lambda): \qquad F_\lambda(f) = \begin{cases} \int_0^1 \phi^+(x,\lambda) f(x) \mathrm{d}x & \text{if } \lambda \in \Gamma^+ \cup \Gamma_0, \\ \int_0^1 \phi^-(x,\lambda) f(x) \mathrm{d}x & \text{if } \lambda \in \Gamma^-, \end{cases}$$

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$$\phi^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \Big[e^{-i\lambda x}(e^{i\lambda} + 2\omega e^{-i\omega\lambda} + 2\omega^2 e^{-i\omega^2\lambda}) + e^{-i\omega\lambda x}(\omega e^{i\omega\lambda} - 2\omega e^{-i\lambda}) + e^{-i\omega^2\lambda x}(\omega^2 e^{i\omega^2\lambda} - 2\omega^2 e^{-i\lambda}) \Big],$$

$$\phi^-(x,\lambda) = \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \Big[e^{-i\lambda x}(2 + \omega^2 e^{-i\omega\lambda} + \omega e^{-i\omega^2\lambda}) + \omega e^{-i\omega\lambda x}(2 - e^{-i\omega^2\lambda}) + \omega^2 e^{-i\omega^2\lambda x}(2 - e^{-i\omega\lambda}) \Big]$$

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Alternative inverse transform for coupled problem *only*:

$$F(\lambda) \mapsto f(x): \qquad f_x^\Sigma(F) = \sum_\sigma \int_{C(\sigma,1/2)} e^{i\lambda x} F(\lambda) \mathrm{d}\lambda, \qquad x \in [0,1].$$

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Then $q(x,t) = f_x(e^{i\lambda^3 t} F_\lambda(q_0))$. The kernels $\phi^\pm$ are:
uncoupled

$$\phi^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \left[e^{-i\lambda x}(\omega e^{-i\omega\lambda} + \omega^2 e^{-i\omega^2\lambda}) - (\omega e^{-i\omega\lambda x} + \omega^2 e^{-i\omega^2\lambda x}) e^{-i\lambda} \right],$$

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Spectral meaning of transform pair

Consider the *spatial differential operator S* defined by

general

$$\text{domain}(S) = \{f \in C^\infty[0,1] : \ f \text{ satisfies the boundary conditions}\},$$
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coupled, **uncoupled**

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Then S is *diagonalized* by the Fourier sine transform & inverse:

$$f_x^{\sin}(F_\lambda^{\sin}(Sf)) = f_x^{\sin}(\lambda^2 F_\lambda^{\sin}(f))$$

ie. the usual spectral theorem for self-adjoint operators.

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Proposition

If IBVP is well-posed, then $\{F_\lambda : \lambda \in \Gamma^+ \cup \Gamma^- \cup \Gamma_0\}$ is a complete family of functionals on the space of compatible initial data.

A new spectral theorem

coupled problem

Theorem

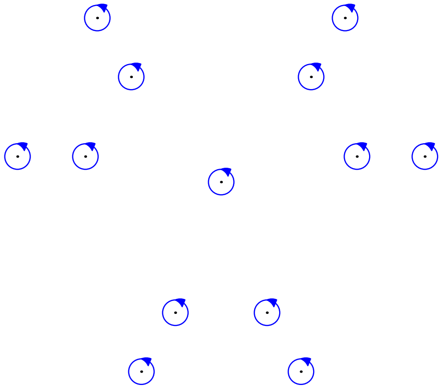
The family of functionals

$$\{F_\lambda : \lambda \in \gamma\}$$

is complete and S is diagonalized by the transform pair in the sense that

$$\int_\gamma e^{i\lambda x} F_\lambda(Sf) d\lambda = \int_\gamma e^{i\lambda x} \lambda^3 F_\lambda(f) d\lambda.$$

A residue calculation yields something like the usual spectral theorem for this non-self-adjoint operator S.



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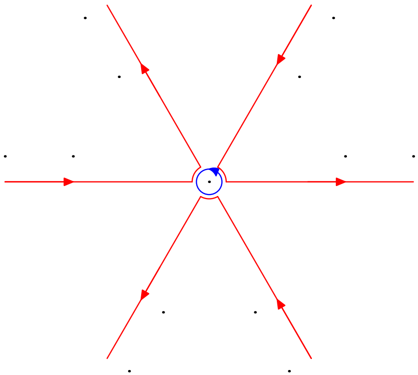

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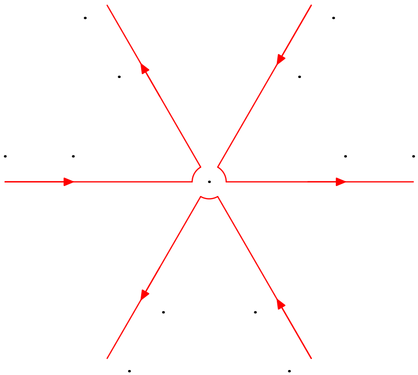

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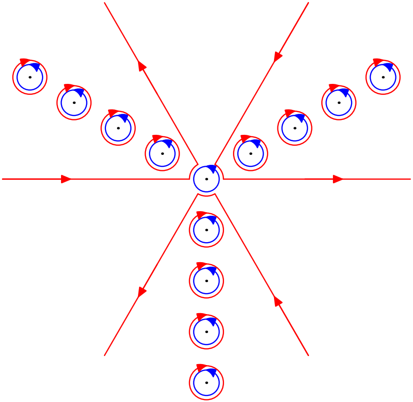

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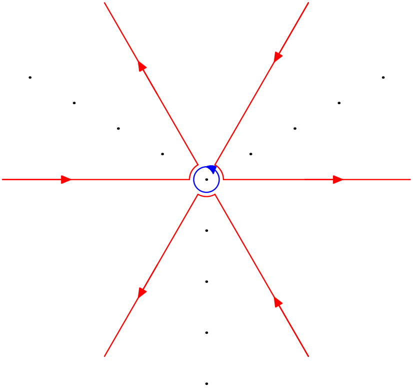

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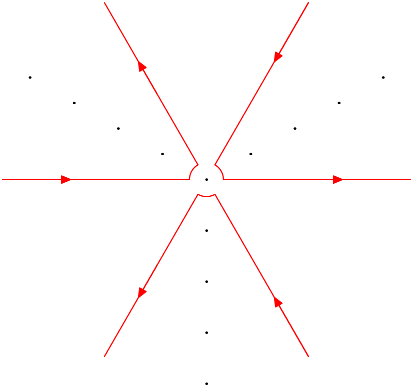

Thank you!

More information on Fokas' unified transform method at
`http://tinyurl.com/fokas-intro`.