Augmented eigenfunctions:
a new spectral object appearing in the integral representation of the solution of linear initial-boundary value problems

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Thanks

Thanks Bernard for the opportunity to speak.

I will present work that is joint with Thanasis Fokas and Beatrice Pelloni.

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Initial-boundary value problem

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$$\begin{array}{llll} \text{PDE} & [\partial_t + \partial_x^3]q(x,t) = 0, & & (x,t) \in (0,1) \times (0,T), \\ \text{IC} & q(x,0) = q_0(x), & & x \in [0,1], \\ \text{BC} & q(0,t) = 0, \quad q(1,t) = 0, \quad q_x(1,t) = 0, & & t \in [0,T]. \end{array}$$

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Problem can be solved by Fokas' unified transform method:

$$q(x,t) = \frac{1}{2\pi}\int_{\Gamma^+} e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda;q_0)}{\Delta(\lambda)}d\lambda + \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda;q_0)}{\Delta(\lambda)}d\lambda,$$

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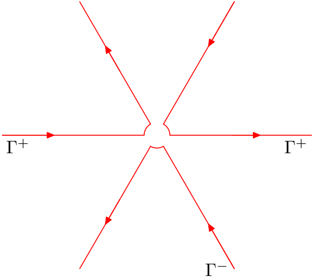
$$\Gamma^\pm = \partial\{\lambda \in \mathbb{C}^\pm : \mathrm{Re}(-i\lambda^3) > 0, |\lambda| > \epsilon\} \; \exists\, \epsilon > 0.$$



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Question

Why do the classical methods fail?
Why does Fokas' method work?

Spectral view of classical techniques

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Assumption: $q(x,t) = \sum_{k\in\mathbb{N}} \xi_k(x)\tau_k(t).$

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We call ξ_k *eigenfunctions* of the differential operator S

$$\mathcal{D}(S) = \{f \in C^\infty[0,1] : f(0) = f(1) = 0\}, \qquad Sf = -f''.$$

Crucially, we have assumed

the eigenfunctions of S are complete in the space of initial data (that is $\mathcal{D}(S)$) and the corresponding series expansions converge.

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(Rough) Definition: Define the operator

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Shkalikov (1976): the eigenfunctions are complete.

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But $F_\lambda(f)$ describe the integral transforms used to solve the Dirichlet problem for the heat equation on the half-line:

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Note: *Completeness* of $\{F_\lambda : \lambda \in \Gamma\}$ means it holds that if $f \in \mathcal{D}(S)$ and $F_\lambda(f) = 0\ \forall\ \lambda \in \Gamma$ then $f = 0$.

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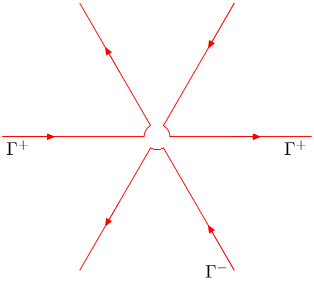

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Then, for S the differential operator giving the spatial part of the linearized KdV problem:

$$\mathcal{D}(S) = \{f \in C^\infty[0,1] : f(0) = f(1) = f'(1) = 0\}, \qquad Sf = if''',$$

Theorem

(F_λ, f_x) is a genuine transform-inverse transform pair on $\mathcal{D}(S)$.

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The forward transforms F_λ can be seen as functionals on $C^\infty[0,1]$.

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F_λ are a complete systems of functionals on $\mathcal{D}(S)$ and the corresponding expansions of each $q_0 \in \mathcal{D}(S)$ converge.

Augmented eigenfunctions

However F_λ are *not generalized eigenfunctions*.

Augmented eigenfunctions

However F_λ are *not generalized eigenfunctions*. Instead, they are a new species of spectral functional:

Definition

Augmented eigenfunctions of S are family of functionals $\{F_\lambda : \lambda \in \Gamma^\pm\}$ on $C^\infty[0,1]$ satisfying

$$F_\lambda(Sf) = \lambda^3 F_\lambda(f) + R_\lambda(f) \qquad \forall\ f \in \mathcal{D}(S),\ \forall\ \lambda \in \Gamma^\pm,$$

$$\int_{\Gamma^\pm} e^{i\lambda x} \frac{1}{\lambda^3} R_\lambda(f) d\lambda = 0 \qquad \forall\ f \in \mathcal{D}(S),\ \forall\ x \in (0,1).$$

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so $\hat{q}(\lambda,t) = e^{-\lambda^2 t}\hat{q}(\lambda,0)$.

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We see that (ii) is a consequence of the stronger property

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something simple + something which disappears when we take the inverse transform.

Moreover the transform-inverse transform pair gives a kind of diagonalization of S:

S is represented with diagonalized inverse.

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Identification of the correct system of augmented eigenfunctions and proofs of convergence are still tied to Fokas' method, but definition is general.
So it may be possible to find other operators with useful augmented eigenfunctions.

Thank you!

Website for Fokas' unified transform method at http://unifiedmethod.azurewebsites.net